# Chapter 7

# Compact Groups.

## 7.1 Haar Measure.

A group is a set $G$ with a binary operation $G \times G \to G$ called multiplication written as $gh \in G$ for $g, h \in G$. It is associative in the sense that $(gh)k = g(hk)$ for all $g, h, k \in G$. A group also has a special element $e$ called the identity that satisfies $eg = ge = g$ for all $g \in G$. It is easy to verify that $e$ is unique. A group also has the property that for each $g \in G$ there is an element $h = g^{-1}$ such that $gh = hg = e$. In general it need not be commutative i.e. $gh \neq hg$. If $gh = hg$ for all $g, h \in G$ the group is called abelian or commutative.

A topological group is a group with a topology such that the binary operation $G \times G \to G$ that sends $g, h \to gh^{-1}$ is continuous. This is seen to be equivalent to the assumption that the operations $G \times G \to G$ defined by $g, h \to gh$ and $G \to G$ defined by $g \to g^{-1}$ are continuous.

We will assume that our group $G$ as a topological space is a compact metric space. Let $\mathcal{B}$ be the class of Borel sets. A measure on $G$ is a nonnegative measure of total mass 1 on $(G, \mathcal{B})$. A left invariant (right invariant) Haar measure on $G$ is a measure $\lambda$ such that $\lambda(g^{-1}A) = \lambda(A)$ $(\lambda(Ag^{-1}) = \lambda(A))$ for all $A \in \mathcal{B}$. Here $g^{-1}A$ is the set of elemnts of the form $g^{-1}h$ with $h \in A$. The set $Ag^{-1}$ is defined similarly.

For any two measures $\alpha, \beta$ on $(G, \mathcal{B})$ the measure $\alpha * \beta$ is defined by

$$\alpha * \beta(A) = \int_G \alpha(Ag^{-1}) d\beta(g) = \int_G \beta(g^{-1}A) d\alpha(g)$$

In terms of integrals

$$\int_G f(k)d(\alpha * \beta)(k) = \int_G \int_G f(gh)d\alpha(g)d\beta(h)$$

**Theorem 7.1.** *The following are equivalent.*

1. *$\lambda$ is a left invariant Haar measure on $G$.*

2. *$\lambda$ is a right invariant Haar measure on $G$.*

3. *$\lambda$ is an idempotent i.e $\lambda * \lambda = \lambda$ and $\lambda(U) > 0$ for every open set $U$.*

The proof will depend on the following

**Lemma 7.2.** *$\alpha * \lambda = \lambda$ for an $\alpha$ with $\alpha(U) > 0$ for all open sets $U \subset G$ if and only if $\lambda(g^{-1}A) = \lambda(A)$ for all $g \in G$ i.e. $\lambda$ is a left invariant Haar measure.*

*Proof.* For any bounded continuous function $u$ on $G$ let us define

$$v(g) = (u * \lambda)(g) = \int_G u(gh)d\lambda(h)$$

Then for any $a \in G$

$$\begin{aligned}\int v(ag)d\alpha(g) &= \int_G \int_G u(agh)d\alpha(g)d\lambda(h)\\ &= \int_G u(ak)d(\alpha * \lambda)(k)\\ &= \int_G u(ak)d\lambda(k)\\ &= v(a)\end{aligned}$$

We can take $a$ to be the element where the maximum of $v$ is attained. Then $v(ag) = v(a)$ for all $g$ in the support of $\alpha$. In particular $v(ag)$ and therefore $v$ is a constant. Hence $\delta_g * \lambda = \lambda$ or $\lambda$ is left invariant. □

*Proof.* (of Theorem). If $\lambda$ is right invariant then $\lambda * \delta_g = \lambda$ for all $g \in G$ and by integrating $\lambda * \lambda = \lambda$ and that implies that $\lambda$ is left invariant as well. We note that any left or right invriant meausre cannot give zero mass to any open set because by compactness $G$ can be covered by a finite number of translates of $U$. □

**Theorem 7.3.** *A left (or right) invariant Haar meausre exists, is unique and is invariant from the right (left) as well.*

*Proof.* Let us start with any $\alpha$ that gives positive mass to every open set and consider

$$\lambda_n = \frac{\alpha + \alpha^2 + \alpha^n}{n}$$

Any weak limit $\lambda$ of $\lambda_n$ satisfies $\alpha * \lambda = \lambda$ and by lemma such a $\lambda$ is left and therefore right invariant. If $\lambda$ is left invariant then $\alpha * \lambda = \lambda$ for any $\alpha$ and if $\alpha$ is also right invariant then $\alpha * \lambda = \alpha$ proving uniqueness. □

## 7.2 Representations of a Group

Given a group $G$, a representation $\pi(g)$ of the group is a continuous mapping $\pi(\cdot)$ of $G$ into nonsingular linear transformations of a finite dimensional complex vector space $V$ such that $\pi(e) = I$ and $\pi(gh) = \pi(g)\pi(h)$ for all $g, h \in G$.

**Theorem 7.4.** *Given a representation $\pi$ of $G$ on a finite dimensional vector space $V$, there is an inner product $< v_1, v_2 >$ on $V$, such that each $\pi(g)$ is a unitary transformation.*

*Proof.* Let $< v_1, v_2 >_0$ be any inner product. We define a new inner product

$$< v_1, v_2 >= \int_G < \pi(h)v_1, \pi(h)v_2 >_0 \, dh$$

where $dh$ is the unique Haar measure on $G$. It is seen that

$$\begin{aligned} < \pi(g)v_1, \pi(g)v_2 > &= \int_G < \pi(h)\pi(g)v_1, \pi(h)\pi(g)v_2 >_0 \, dh \\ &= \int_G < \pi(hg)v_1, \pi(hg)v_2 >_0 \, dh \\ &= \int_G < \pi(h)v_1, \pi(h)v_2 >_0 \, dh \\ &=< v_1, v_2 > \end{aligned}$$

which proves that $\pi(\cdot)$ are unitary with respect to $< \cdot, \cdot >$. □

A representation $\pi(\cdot)$ of $G$ on $V$ is irreducible if there is no proper subspace $W$ of $V$ other than $V$ itself and the subspace $\{0\}$ that is left invariant by $\{\pi(g) : g \in G\}$. Since any finite dimensional representation of $V$ can be made unitary, if $W \subset V$ is invariant so is $W^{\perp}$ and $\pi(\cdot)$ on $V$ is the direct sum of $\pi(\cdot)$ on $W$ and $W^{\perp}$. It is clear that any finite dimensional representation is a direct sum of irreducible representations.

Two unitary representations $\pi_1$ and $\pi_2$ of $G$ on two vector spaces $V_1$ and $V_2$ are said to be equivalent if there is a linear isomorphism $T : V_1 \to V_2$ such that $\pi_2(g)T = T\pi_1(g)$ for all $g \in G$. It is clear that $\pi_1$ and $\pi_2$ are equivalent then either they are both irreducible or neither is. The set of irreducible representations is naturally divided into equivalence classes. We denote them by $\omega \in \Omega$. Each $\omega$ is an equivalence class and $\Omega$ is the set of all equivalence classes.

**Lemma 7.5.** *If $\pi$ is an irreducible representation of $G$ on a finite dimensional vector space $V$, then the inner product that makes the representation unitary is unique up to a scalar multiple.*

*Proof.* If $< \cdot,\cdot >_i$: $i = 1, 2$ are two inner products on $V$ that make $\pi(g)$ unitary for all $g \in G$, then

$$< \pi(g)u, \pi(g)v >_i=< u, v >_i$$

and if we represent by $T$ any unitary isomorphism between the two inner product spaces $\{V, < \cdot,\cdot >_1\}$ and $\{V, < \cdot,\cdot >_2\}$ so that $< u, v >_1=< Tu, Tv >_2$ then

$$< Tu, Tv >_2=< u, v >_1=< \pi(g)u, \pi(g)v >_1=< T\pi(g)u, T\pi(g)v >_2$$

In other words if we denote by $T^*$ the adjoint of $T$, on the inner product space $< \cdot,\cdot >_2$

$$(T^*T)\pi(g) \equiv \pi(g)(T^*T)$$

for all $g \in G$. $T^*T$ is Hermitian and its eigenspaces are left invariant by the $\pi(g)$ that commute with it. These eigenspaces have to be trivial because of irreducibility. That forces $T^*T$ to be a positive mutiple of identity. The two inner products are then essentially the same. □

Given two representations $\pi_1$ and $\pi_2$ on $V_1$ and $V_2$ an intertwining operator from $V_1 \to V_2$ is a linear map $T$ such that $T\pi_1(g) = \pi_2(g)T$ for all $g \in G$.

**Theorem 7.6. Schur's Lemma.** *Given two irreducible representations $\pi_i$ on $V_i$ any intertwining operator $T$, it is either an isomorphism which makes the two representations equivalent or $T = 0$.*

*Proof.* Suppose $T$ has a null space $W \subset V_1$. Then if $u \in W$, $T\pi_1(g)u = \pi_2(g)Tu = 0$ so that $\pi(g)W \subset W$. Either $W = \{0\}$ or $W = U_1$ making $W$ either 0 or one-to-one. By a similar argument the range of $T$ is left invariant by $\pi_2$ making $T$ either 0 or onto. Therefore if $T$ is not an isomorpihsm it is 0. Any isomorphism is essentially a unitary isomorphism. ☐

## 7.3 Representations of a Compact group.

A natural infinite dimensional representation of a compact group is the (left) regular representation $L_g$ on $L_2(G, dg)$ defined by $(L_g u)(h) = u(g^{-1}h)$ satisfying $L_{g_1}L_{g_2} = L_{g_1 g_2}$. From the invariance of the Haar measure $L_g$ is unitary. First we prove some facts regarding finite dimensional representations of compact groups.

**Theorem 7.7.** *Let $\pi$ be a finite dimnsional irreducible representation of $G$ in a space of dimension $d$. Then there is subspace of dimension $d^2$ in $L_2(G, dg)$ that is invariant under both left and right regular representations and either one on this subspace decomposes into $d$ copies of $\pi$. The representation of this type, i.e. any equivalent representation does not occur in the orthogonal complement of this $d^2$ dimensional subspace.*

*Proof.* Let $\pi$ be a representation of $G$ in a finite dimensional space $V$. Pick a basis for $V$ and represent $\pi(g)$ as a unitary matrix $\{t_{i,j}(g)\}$. The functions $t_{i,j}(\cdot)$ are continuous and are in $L_2(G, dg)$. Let us see what the left regular representation $L_h$ does to them.

$$t_{i,j}(hg) = [\pi(h)\pi(g)]_{i,j} = \sum_r t_{i,r}(h)t_{r,j}(g)$$

which is the same as

$$L_h t_{i,j}(\cdot) = \sum_r t_{i,r}(h)t_{r,j}(\cdot)$$

or for each $j$ the space spanned by $\{t_{r,j}(\cdot) : 1 \le r \le d\}$ is invariant under $L_h$ and transforms like $\pi$. Similarly under $R_{h^{-1}}$, the rows $\{t_{j,r}(\cdot) : 1 \le r \le d\}$ will again transform like $\pi$. If we can show that $\{t_{i,j}(\cdot)\}$ are linearly independent then $d^2$ dimensional space will transform like $d$ copies of $\pi$ under $L_h$ and $R_h$. Consider two representations $\pi_1$,$\pi_2$ that are irreducible on $V_1, V_2$ and vectors $u_1, v_1$ and $u_2, v_2$ in their respective spaces. Then for fixed $v_1, v_2$

$$\int_G <\pi_1(g)u_1, v_1>_1 \overline{<\pi_2(g)u_2, v_2>_2} dg = <Bu_1, u_2>$$

defines an operator from $V_1 \to V_2$ and a calculation

$$\begin{aligned}
<B\pi_1(h)u_1, u_2> &= \int_G <\pi_1(g)\pi_1(h)u_1, v_1>_1 \overline{<\pi_2(g)u_2, v_2>_2} dg \\
&= \int_G <\pi_1(gh)u_1, v_1>_1 \overline{<\pi_2(g)u_2, v_2>_2} dg \\
&= \int_G <\pi_1(g)u_1, v_1>_1 \overline{<\pi_2(gh^{-1})u_2, v_2>_2} dg \\
&= \int_G <\pi_1(g)u_1, v_1>_1 \overline{<\pi_2(g)\pi_2(h^{-1})u_2, v_2>_2} dg \\
&= \int_G <\pi_1(g)u_1, v_1>_1 \overline{<\pi_2(g)\pi_2^*(h)u_2, v_2>_2} dg \\
&= <Bu_1, \pi_2^*(h)u_2> \\
&= <\pi_2(h)Bu_1, u_2>
\end{aligned}$$

Showing that $B$ intertwines $\pi_1$ and $\pi_2$. If the representations are inequivalent then $B = 0$. Otherwise, $B = c(v_1, v_2)T$ where $T$ is the isomorphism between $V_1$ and $V_2$ that intertwines $\pi_1$ and $\pi_2$ and a further calculation of the same nature shows that $c(v_1, v_2) = c <Tv_1, v_2>_2$. Therefore

$$\int_G <\pi_1(g)u_1, v_1>_1 \overline{<\pi_2(g)u_2, v_2>_2} dg = c <Tu_1, u_2>_2 <Tv_1, v_2>_2$$

where $c = 0$ for inequivalent representations. In the equivalent case taking $\pi \equiv \pi_1 \equiv \pi_2$, $V = V_1 = V_2$ and $T = I$,

$$\int_G <\pi(g)u_1, v_1> \overline{<\pi(g)u_2, v_2>} dg = c <u_1, u_2>_2 <v_1, v_2>$$

To calculate $c$ we take $u = u_1 = u_2$ and $v = v_1 = v_2$

$$c\|u\|^2\|v\|^2 = \int_G | <\pi(g)u, v> |^2 dg = \int_G |\sum_{i,j} t_{i,j}(g) u_i v_j|^2 dg$$

Thus
$$\int_G t_{i,j}(g)\overline{t_{rs}(g)}dg = c\delta_{ir}\delta_{js}$$
On the other hand
$$\sum_{i,j} |t_{i,j}|^2 \equiv d$$
so that $c = \frac{1}{d}$. The character of the represenation defined as $\chi_\pi(g) = \text{trace } \pi(g)$ is independent of concrete vector space used for the representation and $\|\chi_\pi\|_{L_2(G,dg)} = 1$. If $\pi$ occured again in the orthogonal complement of the $d$ dimensional space, then that space would have to contain $\chi_\pi$ again and that is not possible. □

Now we show that there are lots of finite dimensional representations.

**Theorem 7.8.** *Any right trnaslation $R_k$ defined by $(R_k u)(h) = u(hk)$ commutes with $L_g$. There are compact self adjoint operators that commute with the family $\{L_g\}$. Hence the representation $L_g$ on $L_2(G, dg)$ decomposes into a sum of irreducible finite dimensional represenations. In particular a compact group has sufficiently many irreducible finite dimensional representations. More precisely given $g \neq e$ there is one for which $\pi(g) \neq I$.*

*Proof.* Clearly
$$(R_k L_g u)(h) = (R_k L_g u)(h) = u(g^{-1}hk)$$
The integral operators
$$(Tu)(h) = \int_G u(hg)\tau(g)dg = \int_G u(g)\tau(h^{-1}g)dg$$
commute with $L_g$ and are compact (in fact Hilbert-Schmidt) and self adjoint provided
$$\int_G \int_G |\tau(h^{-1}g)|^2 dg dh < \infty$$
and for all $k$,
$$\tau(k^{-1}) = \overline{\tau(k)}$$
The eigenspaces of $T$ of finite dimension provide lots of finite dimensional representations that can then be split up into irreducible pieces. We can check that the only possible infinite dimensional piece is the null space of $T$. Let us write $L_2(G, dg) = \oplus_j V_j \oplus V_\infty$ as the direct sum of finite dimensional pieces

that are invariant under both $L_g$ and $R_g$ if possible an infinite dimensional piece $V_\infty$ that is also invariant under $L_g$ and $R_g$ and has no such nontrivial finite dimensional invariant subspace. The earlier argument of convolution by $\tau$ can be repeated on $V_\infty$ and produces a finite dimensional $L_g$ invariant subspace, which is a contradiction unless such a convolution is identically 0 on $V_\infty$ for all $\tau$. This is seen to be impossible. □

The character $\chi_\pi(g)$ determine $\pi$ completely and for inequivalent representations they are orthogonal. The characters have the additional property that $\chi_\pi(hgh^{-1}) = \chi_\pi(g)$.

**Theorem 7.9.** *Any function $u(g) \in L_2(G, dg)$ such that $u(hgh^{-1}) = u(g)$ i.e. $L_h u = R_h u$ for all $h \in G$ is spanned by the characters.*

*Proof.* We need to show that if any such $u$ is orthogonal to $\chi_\pi$ it is also orthogonal to all the matrix elements.

$$\begin{aligned}
\int_G u(g)\overline{t_{r,s}(g)}dg &= \int u(hgh^{-1})\overline{t_{r,s}(g)}dg \\
&= \int_G u(g)\overline{t_{r,s}(h^{-1}gh)}dg \\
&= \int_G u(g)\overline{[\pi^*(h)\pi(g)\pi(h)]_{r,s}}dg \\
&= \int_G u(g)\sum_{i,j} t_{i,r}(h)\overline{t_{i,j}(g)t_{j,s}(h)}dg \\
&= \int_G\int_G u(g)\overline{t_{i,j}(g)}t_{i,r}(h)\overline{t_{j,s}(h)}dgdh \\
&= \sum_{i,j}\frac{1}{d}\delta_{i,j}\delta_{r,s}\int_G u(g)\overline{t_{i,j}(g)}dg \\
&= \frac{1}{d}\delta_{r,s}\int_G u(g)\overline{\chi_\pi(g)}dg = 0
\end{aligned}$$

□